

THE ACTION OF AFFINE DIFFEOMORPHISMS ON THE RELATIVE COHOMOLOGY OF ABELIAN COVERS OF THE FLAT PILLOWCASE

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ABSTRACT. We calculate the action of the group of affine diffeomorphisms on the relative cohomology of square-tiled surfaces that are normal abelian covers of the flat pillowcase, and as an application, answer a question raised by Smillie and Weiss.

1. INTRODUCTION

In a paper in preparation by Smillie and Weiss on horocycle orbit closures they ask for the existence of a square-tiled surface M with cone point sets Σ constructed as a normal abelian branched cover of the flat pillowcase, such that there is a direct sum decomposition $H^1(M, \Sigma; \mathbb{C}) = N \oplus H$ preserved by the action of the group of orientation preserving affine diffeomorphisms, there is a positive or negative definite hermitian norm on N invariant under the affine diffeomorphism group action, and the affine diffeomorphism group action on N does not factor through a discrete group. We will construct such examples in Section 6.

In answering this question, we give a comprehensive treatment on relative cohomology of branched abelian covers of the flat pillowcase, the affine diffeomorphism group action on it, as well as the invariant subspaces and invariant Hermitian form under this action. This problem is related to the monodromy of the hypergeometric functions which dates back to Euler and is outlined in [DM86]. Wright [Wri12] described the Hodge form and invariant direct sum decomposition on the absolute cohomology of such surfaces under the action of a subgroup of the Veech group, and calculated the Lyapunov exponents of this action by showing that the projectivization of the group action factors through a triangle group. Forni-Matheus-Zorich [FMZ11], Bouw-Möller [BM10], Deligne-Mostow [DM86], McMullen [McM13] and Eskin-Kontsevich-Zorich [EKZ10] have also done similar computations in different contexts. Matheus and Yoccoz [MY09] calculated the action of the full affine group on relative cohomology for two specific abelian branched covers. By modifying some of the ideas in their articles, as well as some arguments similar to [DM86] and [Thu98], we are able to describe the action of the affine group on relative cohomology and establish the existence of abelian covers as required by Smillie and Weiss's paper. Hubert-Schmithüsen [HSb] also gave a proof of the non-discreteness in some cases through Lyapunov exponents and Galois conjugate. The author thanks his thesis advisor John Smillie for suggesting the problem and many helpful conversations.

The existence of examples answering the question of Smillie and Weiss follows from a decomposition of cohomology into invariant components, which is done in Theorem 3.1, a signature calculation of the Hodge form on each component, and a discreteness criteria. Here we give an alternative, self-contained treatment. We will give a description of the affine diffeomorphism group of these surfaces in section 2. In section 3, we describe the action of affine diffeomorphism group on relative cohomology and show the existence of direct sum decomposition. In section 4, we calculated the signature of the Hodge Hermitian form. The reader is warned that our definition of Hodge Hermitian form is different from other definitions in the literature. In section 5 we described a useful subgroup of affine diffeomorphism group to work with. In section 6 we construct examples that answer the question of Smillie and Weiss. If we only need to construct certain examples, it can also be done with discreteness criteria and signature calculation in [DM86].

We will now set up some notation to describe normal branched covers of the pillowcase. Let P be the unit flat pillowcase with four cone points z_1, z_2, z_3 and z_4 of cone angle π as follows:

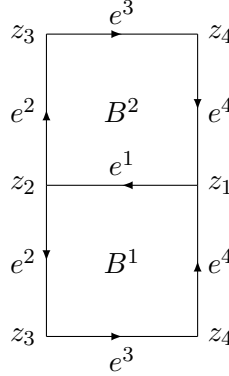


FIGURE 1.

Let G be a finite group and $\mathbf{g} = (g_1, \dots, g_4) \in G^4$ a 4-tuple of elements in G such that $g_1 g_2 g_3 g_4 = 1$. Let $M = M(G, \mathbf{g})$, be the connected normal branched cover of P branching at z_1, \dots, z_4 , with deck transformation group G acting on the left. The loop l_j around z_j in counter-clockwise direction on P based in B^1 lifts to a path from the preimage of B^1 in the g -th sheet of the cover to the preimage of B^1 in the gg_j -th sheet. In other words, \mathbf{g} gives a group homomorphism from

$$\pi_1(P - \{z_1, z_2, z_3, z_4\}) = \langle l_1, l_2, l_3, l_4 | l_1 l_2 l_3 l_4 = 1 \rangle$$

to G . Here the homomorphism defined by \mathbf{g} sends l_j to $g_j \in G$. The connectedness of M is equivalent to the condition that $\{g_1, \dots, g_4\}$ generate G . Let Σ denote the set of preimages of all points z_j , $j = 1, \dots, 4$. The surface M has a half translation structure induced by the half translation structure on P . Let $\mathbf{Aff}(M, \Sigma)$ denote the group of orientation preserving

affine diffeomorphisms from M to itself that sends Σ to Σ . When the orders of g_j are all even, all the holonomies are translations and M is a translation surface. When the order of g_j is 2, the corresponding vertex has cone angle 2π . When none of the orders of g_j is 2, Σ consists of actual cone points of M , in which case \mathbf{Aff} is the affine diffeomorphism group.

The decomposition of P into two squares in figure 1 induces a cell decomposition on $M(G, \mathbf{g})$, which can be described as $|G|$ -copies of pair of squares labeled by elements in G as B_g^1, B_g^2 , that are glued together by identifying edges e_g^j and $e_{g'}^{j'}$ when $j = j'$ and $g = g'$, so that the directions indicated by the arrows match:

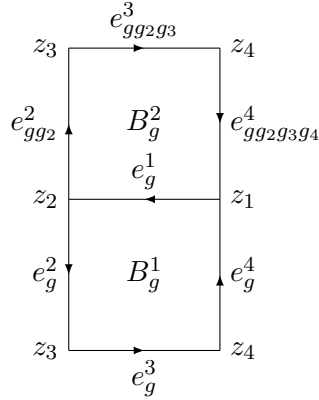


FIGURE 2.

For example, in our notation the Wollmilchsau [For06][HSa] is $M(\mathbb{Z}/4, (1, 1, 1, 1))$, can be presented as the union of the following squares with indicated glueings :

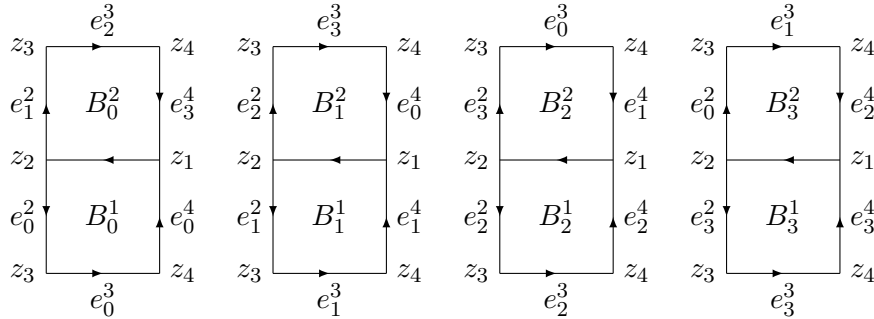


FIGURE 3.

As another example, let $G = \mathbb{Z}/3$ and $\mathbf{g} = (0, 1, 1, 1)$. In this case $M = M(G, \mathbf{g})$ is a half translation surface and the gluing is as follows:

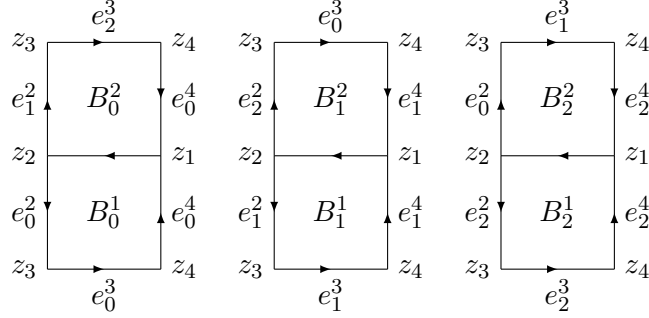


FIGURE 4.

Now we describe the action of the deck group G on $M(G, \mathbf{g})$. An element $h \in G$ sends B_g^k to B_{hg}^k and e_g^k to e_{hg}^k . The deck group action induces a right G -action on $H^1(M, \Sigma; \mathbb{C})$ that makes it a right G -module.

2. AFFINE DIFFEOMORPHISMS

From now on we assume that G is abelian, though many of our arguments work for any finite group. At the end of this section we will point out the modification required in the non-Abelian case.

We calculate $\mathbf{Aff} = \mathbf{Aff}(M(G, \mathbf{g}))$ in a way inspired by the coset graph description used in [Sch04]. One distinction is that we consider the whole affine diffeomorphism group while [Sch04] considers only the Veech group. Fixing G , let V be the set of all 4-tuples $\mathbf{h} = (h_1, h_2, h_3, h_4)$ such that $\{h_1, h_2, h_3, h_4\}$ generates G and $h_1 h_2 h_3 h_4 = 1$, each is associated with a square-tiled surface $M(G, \mathbf{h})$ which is equipped with a cell decomposition labeled as in figure 2. By construction, an element F in \mathbf{Aff} induces an automorphism of the deck group G by $g \mapsto FgF^{-1}$, i.e. there is a group homomorphism $\mathbf{Aff} \rightarrow \text{Aut}(G)$. We denote the kernel of this homomorphism as Γ . Because $\text{Aut}(G)$ is a finite group, Γ is a subgroup of \mathbf{Aff} with finite index.

We will show that all orientation preserving affine diffeomorphisms between various $M(G, \mathbf{h})$ that preserves Σ are compositions of a finite affine diffeomorphisms, which we call basic affine diffeomorphisms, which we will describe below. In our discussion we will be dealing with both translation surfaces and half translation surface surfaces. It will be convenient to view the derivative of an affine diffeomorphism as an element of $PGL(2, \mathbb{R}) = GL(2, \mathbb{R})/\{\pm I\}$. We will call an affine translation diffeomorphism a half translation equivalence when its derivative is 1 in $PGL(2, \mathbb{G})$.

Now we define four of the five classes of the basic affine diffeomorphisms:

- (i) Rotation: $t_{(h_1, h_2, h_3, h_4)}, h_j \in G$ is a map from $M(G, (h_2, h_3, h_4, h_1))$ to $M(G, (h_1, h_2, h_3, h_4))$ and sends B_e^1 of $M(G, (h_2, h_3, h_4, h_1))$ to B_e^1 of $M(G, (h_1, h_2, h_3, h_4))$ by rotating counterclockwise by $\pi/2$.
- (ii) Deck transformation: $r_{g, \mathbf{h}}, g \in G, \mathbf{h} \in G^4$ is the deck transformation g in $M(G, \mathbf{h})$. Its derivative is 1.
- (iii) Interchange of B^1 and B^2 : $f_{(h_1, h_2, h_3, h_4)}, h_j \in G$ is a map from $M(G, (h_2, h_1, h_1^{-1}h_4h_1, h_2h_3h_2^{-1}))$ to $M(G, (h_1, h_2, h_3, h_4))$ which interchanges B_g^1 and B_g^2 by a rotation of π .
- (iv) Relabeling: $m_\psi, \psi \in \text{Aut}(G)$ is a map from $M(G, \mathbf{h})$ to $M(G, \psi(\mathbf{h}))$ and sends B_g^j to $B_{\psi(g)}^j$. Its derivative is 1.

We claim that any half translation equivalence from $M(G, \mathbf{h})$ to $M(G, \mathbf{h}')$ can be written as composition of basic affine diffeomorphisms t^2, r, f and m . Because by our assumption they preserve Σ , they can be seen as a permutation of unit squares that tiled M and M' . More precisely, any half translation equivalence is completely determined by the following data: i) the induced automorphism ψ of deck group, ii) a number $j = 1$ or 2 , i.e. whether or not we interchange B^1 and B^2 , an element $g \in G$, such that $F_0(B_e^1) = B_g^j$, and whether $F_0^{-1}(e_g^1)$ is e_e^1 or e_e^3 . Here e is the unit in G . We can now use m to deal with ψ , then use r and f to send B_e^1 to B_g^j , and if needed precompose with t^2 .

For general orientation preserving affine diffeomorphism F , DF will be in $PSL(2, \mathbb{Z})$. We add another class of basic affine diffeomorphisms:

- (v) Shearing: $s_{(h_1, h_2, h_3, h_4)}, h_j \in G$ is a map from $M(G, (h_1h_2h_1^{-1}, h_1, h_3, h_4))$ to $M(G, (h_1, h_2, h_3, h_4))$ that sends e_1^3 to e_1^3 and has derivative $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Because the derivative of s and t generate $PSL(2, \mathbb{Z})$, by successively composing with s and t we can reduce to the case when derivative is identity. Hence any affine diffeomorphisms between $M(G, \mathbf{h}), \mathbf{h} \in V$ that sends Σ to Σ , or more specifically, any element in \mathbf{Aff} , is a composition of the five classes of maps described above. Because m commutes with other 4 classes of diffeomorphisms, i.e.

$$\begin{aligned}
 m_\psi t_{\mathbf{h}} &= t_{\psi(\mathbf{h})} m_\psi \\
 m_\psi r_{g, \mathbf{h}} &= r_{\psi(g), \psi(\mathbf{h})} m_\psi \\
 m_\psi f_{\mathbf{h}} &= f_{\psi(\mathbf{h})} m_\psi \\
 m_\psi s_{\mathbf{h}} &= s_{\psi(\mathbf{h})} m_\psi
 \end{aligned}$$

any $F \in \mathbf{Aff}$ can be written as $F = F_1 m_\psi$ where F_1 is a composition of t, s, r and f , while ψ is the automorphism of deck group induced by F . Hence, elements in Γ can be written as successive compositions of t, r, f and s .

As in [Sch04], consider the directed graph D with vertex set V , each element $\mathbf{h} \in V$ corresponding to a surface $M(G, \mathbf{h})$, the edges in the graph corresponding to basic affine

diffeomorphisms. Paths starting and ending at $M(G, \mathbf{g})$ correspond to elements in \mathbf{Aff} . Now the fact that any affine diffeomorphism is a successive composition of t , s , r , f , and m means that the map from the set of such paths to \mathbf{Aff} is surjective. Similarly, let D_0 be graph D with those edges corresponding to m removed, then the set of paths starting and ending at \mathbf{g} in D_0 maps surjectively to Γ .

Consider the example $G = \mathbb{Z}/6$, $\mathbf{g} = (1, 1, 1, 3)$. This example is the Ornithorynque[FM08]. In the following figure we give the connected component of D that contains \mathbf{g} , with loops corresponding to deck transformation (i.e. all the r arrows) omitted:

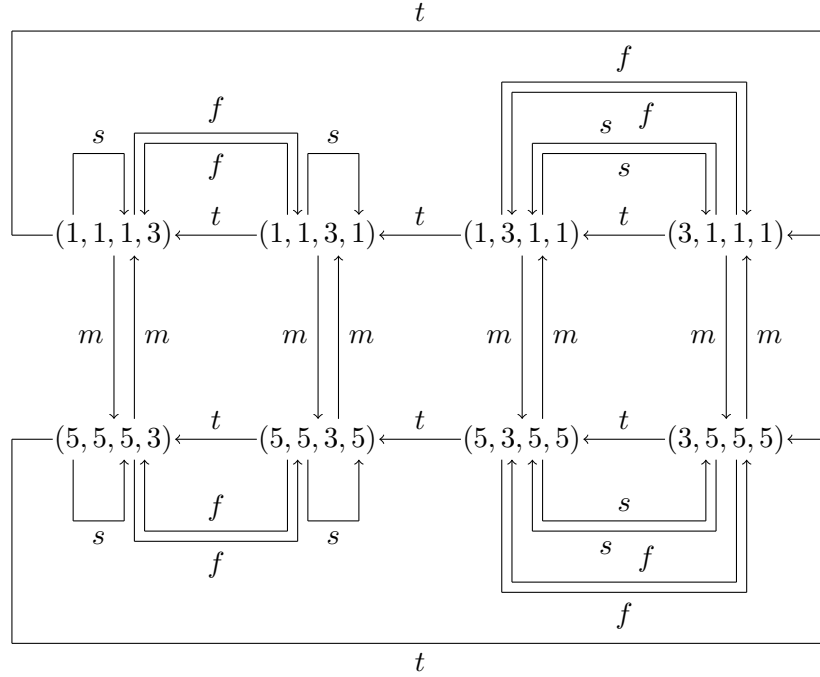


FIGURE 5.

This method of calculating \mathbf{Aff} does not use the fact that G is abelian in any important way. When G is non-abelian, we can define Γ in a slightly different way. In general, we let Γ be the elements in $\mathbf{Aff}(M)$ that induce an inner automorphism on G , then elements in Γ are compositions of t , s , r , f as well as m_ψ where ψ is an element of an inner automorphism of G .

3. INVARIANT DECOMPOSITION OF RELATIVE COHOMOLOGY

We start by giving a direct sum decomposition of $H^1(M(G, \mathbf{g}), \Sigma)$, and calculate the dimension of the summands as well as the action of the affine diffeomorphisms on them.

Theorem 3.1. *Let Δ be the set of irreducible representations of a finite abelian group G , i.e. homomorphisms from G to \mathbb{C}^* . Let $M = M(G, \mathbf{g})$, $\Sigma \subset M$ be defined as in section 1, then*

$$H^1(M, \Sigma; \mathbb{C}) = \bigoplus_{\rho \in \Delta} H^1(\rho)$$

where $H^1(\rho)$ are invariant subspaces under $\Gamma \subset \mathbf{Aff}$ as defined in section 1, and are permuted by the action of \mathbf{Aff} . The dimension of $H^1(\rho)$ is 3 if ρ is the trivial representation, 2 if otherwise.

A compatible splitting for the absolute cohomology $H^1(M)$ was described in [Wri12]. In the case of absolute cohomology the summands can have dimensions 0, 1 or 2. $H^1(\rho)$ can also be described as cohomology with twisted coefficients as in [DM86], [Thu98].

Proof. Consider the relative cellular cochain complex

$$0 \rightarrow C^1(M, \Sigma; \mathbb{C}) \rightarrow C^2(M, \Sigma; \mathbb{C})$$

We can identify $C^1(M, \Sigma; \mathbb{C})$ with $(\mathbb{C}[G])^4$ as a right- G module by writing $m \in C^1(M, \Sigma; \mathbb{C})$ as

$$\left(\sum_g m(e_g^1)g^{-1}, \sum_g m(e_g^2)g^{-1}, \sum_g m(e_g^3)g^{-1}, \sum_g m(e_g^4)g^{-1} \right) \in (\mathbb{C}[G])^4$$

identify $C^2(M, \Sigma; \mathbb{C})$ with $(\mathbb{C}[G])^2$ as a right- G module by writing $n \in C^2(M, \Sigma; \mathbb{C})$ as

$$\left(\sum_g n(B_g^1)g^{-1}, \sum_g n(B_g^2)g^{-1} \right) \in (\mathbb{C}[G])^2$$

then the coboundary map from C^1 to C^2 is

$$(1) \quad d^1(a, b, c, d) = (a + b + c + d, a + g_2b + g_2g_3c + g_2g_3g_4d)$$

Hence:

$$(2) \quad H^1(M, \Sigma; \mathbb{C}) = \{(a, b, c, d) \in (\mathbb{C}[G])^4 : a + b + c + d = a + g_2b + g_2g_3c + g_2g_3g_4d = 0\}$$

Because $\mathbb{C}[G]$ is semisimple [Ser77], it splits into simple algebras $\mathbb{C}[G] = \bigoplus_{\rho \in \Delta} D_\rho$, where D_ρ is the simple subalgebras of $\mathbb{C}[G]$ corresponding to irreducible representation ρ . The splitting of the algebra gives a splitting of the claim complex $0 \rightarrow C^1 \rightarrow C^2$, hence a splitting of the cohomology:

$$(3) \quad H^1(M, \Sigma; \mathbb{C}) = \bigoplus_{\rho \in \Delta} H^1(\rho), H^1(\rho) = \{(a, b, c, d) \in D_\rho^4 : d^1(a, b, c, d) = 0\}$$

The image of d^1 in $C^2(M, \Sigma; \mathbb{C})$ is

$$(1, 1)\mathbb{C}[G] \oplus (0, 1)M \subset (\mathbb{C}[G])^2 = C^2(M, \Sigma; \mathbb{C})$$

where M is the right ideal generated by $\{g_2 - 1, g_2g_3 - 1, g_2g_3g_4 - 1\}$. This is because

$$\begin{aligned} d^1(a, b, c, d) &= (a + b + c + d, a + g_2b + g_2g_3c + g_2g_3g_4d) \\ &= (a + b + c + d, a + b + c + d) + (0, (g_2 - 1)b) + (0, (g_2g_3 - 1)c) + (0, (g_2g_3g_4 - 1)d) \\ &= (a + b + c + d)(1, 1) + ((g_2 - 1)b + (g_2g_3 - 1)c + (g_2g_3g_4 - 1)d)(0, 1) \end{aligned}$$

Now we show that $\mathbb{C}[G] = \mathbb{C} \oplus M$. Because $(1 - a) + (1 - b)a = 1 - ba$, if g is a product of elements in $\{g_2, g_2g_3, g_2g_3g_4\}$ then $1 - g \in M$. Also, because M is connected, $\{g_2, g_2g_3, g_2g_3g_4\}$ generates G , hence M is generated by all elements of the form $1 - g$ for any $g \in G$, therefore $\mathbb{C}[G] = \mathbb{C} \oplus M$, where \mathbb{C} is the trivial sub-algebra generated by $\sum_{g \in G} g$ [Ser77]. Because $\mathbb{C}[G]$ is semisimple,

$$H^1(M, \Sigma; \mathbb{C}) = \ker(d^1) \rightarrow C^1(M, \Sigma; \mathbb{C}) \rightarrow \text{im}(d^1)$$

splits, hence we have

$$(4) \quad H^1(M, \Sigma; \mathbb{C}) = (\mathbb{C}[G])^4 / (\mathbb{C}[G] \oplus M) = (\mathbb{C}[G])^2 \oplus \mathbb{C}$$

Therefore, as G -module $H^1(\rho) \cong \mathbb{C}^3$ when ρ is the trivial representation, $H^1(\rho) \cong D_\rho^2$ if otherwise. Because G is abelian, $\dim_{\mathbb{C}} D_\rho = 1$, so $\dim_{\mathbb{C}} H^1(\rho) = 3$, if ρ is trivial, $\dim_{\mathbb{C}} H^1(\rho) = 2$, otherwise.

In the previous section we describe elements in **Aff** as compositions of elementary affine diffeomorphisms $t_{\mathbf{h}}, s_{\mathbf{h}}, r_{g,\mathbf{h}}, f_{\mathbf{h}}$ and m_ψ , and elements in Γ as compositions of elementary affine diffeomorphisms $t_{\mathbf{h}}, s_{\mathbf{h}}, r_{g,\mathbf{h}}$ and $f_{\mathbf{h}}$. We will show the invariance of $H^1(\rho)$ under Γ by explicitly describing the action of elementary affine diffeomorphisms. The induced map of $t_{\mathbf{h}}, s_{\mathbf{h}}, r_{g,\mathbf{h}}, f_{\mathbf{h}}$ from $H^1(M(G, \mathbf{h}), \Sigma; \mathbb{C})$ to some $H^1(M(G, \mathbf{h}), \Sigma; \mathbb{C})$ are as follows:

$$(5) \quad t_{\mathbf{h}}^*([a, b, c, d]) = [b, c, d, a]$$

$$(6) \quad s_{\mathbf{h}}^*([a, b, c, d]) = [-h_1a, a + b, c, d + h_1a]$$

$$(7) \quad r_{g,\mathbf{h}}^*([a, b, c, d]) = [ag, bg, cg, dg]$$

$$(8) \quad f_{\mathbf{h}}^*([a, b, c, d]) = [-a, -h_2h_3h_4d, -h_2h_3c, -h_2b]$$

From equation (3) we know that they all preserve decomposition $H^1(*, \Sigma; \mathbb{C}) = \bigoplus_{\rho} H^1(\rho)$, hence all summands $H^1(\rho)$ are invariant under Γ .

Furthermore, m_ψ is a diffeomorphism from $M(G, \psi^{-1}\mathbf{h})$ to $M(G, \mathbf{h})$, and the map it induced from $H^1(M(G, \mathbf{h}), \Sigma; \mathbb{C})$ to $H^1(M(G, \psi(\mathbf{h})), \Sigma; \mathbb{C})$ is

$$\begin{aligned} m_\psi^*([\sum_{g \in G} a_g g, \sum_{g \in G} b_g g, \sum_{g \in G} c_g g, \sum_{g \in G} d_g g]) \\ = [\sum_{g \in G} a_g \psi^{-1}(g), \sum_{g \in G} b_g \psi^{-1}(g), \sum_{g \in G} c_g \psi^{-1}(g), \sum_{g \in G} d_g \psi^{-1}(g)] \end{aligned}$$

which, according to equation (3), would send $H^1(\rho)$ to $H^1(\psi^{-1}\rho)$. In other words, elements in **Aff** permute $H^1(\rho)$. □

Remark 1. In certain situations $\Gamma = \mathbf{Aff}$. This happens when the g_j are all of different order, or when G is \mathbb{Z}/n , $n \geq 4$ and $\mathbf{g} = (1, 1, 1, n-3)$. In these cases $H^1(\rho)$ are all invariant under \mathbf{Aff} . Our argument here is similar to, but not completely the same as those used in [MY09].

4. THE SIGNATURE OF THE HODGE FORM

Now we define and calculate the signature of an invariant Hermitian form on $H^1(\rho)$ as in [Thu98] and [DM86].

The Hodge form A_G , or area form as in [Thu98], on $H^1(M, \Sigma; \mathbb{C})$ is defined as $\frac{1}{2i}$ of the cup product with coefficient pairing $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} : (z, z') \mapsto z\overline{z'}$ on (M, Σ)

$$H^1(M, \Sigma; \mathbb{C}) \times H^1(M, \Sigma; \mathbb{C}) \xrightarrow{\sim} H^2(M, \Sigma; \mathbb{C}) = \mathbb{C}$$

In other words,

$$(9) \quad \begin{aligned} A_G([a, b, c, d], [a', b', c', d']) &= \frac{1}{4i}((b, a')_G - (a, b')_G + (d, c')_G - (c, d')_G \\ &\quad - (h_2 b, a')_G + (a, h_2 b')_G - (h_2 h_3 h_4 d, h_2 h_3 c')_G + (h_2 h_3 c, h_2 h_3 h_4 d')_G) \end{aligned}$$

where $(\cdot, \cdot)_G$ is the positive definite Hermitian norm on $\mathbb{C}[G]$ defined as

$$\left(\sum_g a_g g^{-1}, \sum_g b_g g^{-1}\right)_G = \sum_g a_g \overline{b_g}$$

Alternatively, if elements in $H^1(M, \Sigma; \mathbb{C})$ are represented by closed differential forms, A_G can be written as $A_G(\alpha, \beta) = \frac{1}{2i} \int_M \alpha \wedge \overline{\beta}$.

By definition A_G is invariant under the Γ -action. Furthermore, from (9) and the fact that different D_ρ are orthogonal under $(\cdot, \cdot)_G$, we know that $H^1(\rho)$ for different representation ρ are orthogonal to each other under A_G . When ρ is the trivial representation, $A_G = 0$ on $H^1(\rho)$.

Now we assume ρ to be a non-trivial representation. Because we will deal with Hermitian forms that may be degenerate, we denote the signature of a Hermitian form as (n_0, n_+, n_-) , where n_0, n_+, n_- are the number of 0, positive and negative eigenvalues respectively.

We will prove the following theorem:

Theorem 4.1. *The signature of the area form A_G on $H^1(\rho)$, $\rho \neq 1$ is $(n_0, \frac{\theta_2}{2\pi} - 1, \frac{\theta_1}{2\pi} - 1) = (4 - \frac{\theta_1 + \theta_2}{2\pi}, \frac{\theta_2}{2\pi} - 1, \frac{\theta_1}{2\pi} - 1)$, where $\theta_1 = \sum_{j=1}^4 \arg(\rho(g_j))$, $\theta_2 = \sum_{j=1}^4 \arg(\rho(g_j)^{-1})$. The number n_0 is also the number of indices j such that $\rho(g_j) = 1$.*

Proof. In the case when $\rho(g_1 g_2) = \rho(g_2 g_3) = 1$, $a = c$, $b = d$, and A_G is $2|G|$ times the area of parallelogram with side a and b , i.e. proportional to the cross product of two vectors on the complex plane, which has signature $(0, 1, 1) = (4 - \frac{\theta_1 + \theta_2}{2\pi}, \frac{\theta_2}{2\pi} - 1, \frac{\theta_1}{2\pi} - 1)$.

Now we consider the case when $\rho(g_1g_2) \neq 1$ or $\rho(g_2g_3) \neq 1$. Without losing generality we assume $\rho(g_1g_2) \neq 1$. From (1), (3) we know that any $(a, b, c, d) \in H^1(\rho)$ satisfies

$$(10) \quad a + b + c + d = 0$$

$$(11) \quad a + \rho(g_2)b + \rho(g_2g_3)c + \rho(g_2g_3g_4)d = 0$$

Because $\rho(g_3g_4) = \rho(g_1g_2)^{-1} \neq 1$, we can solve (b, d) from these two equations as linear functions of (a, c) , i.e. rewrite these equations can be written as $(b, d) = (a, c)A$ where A is a 2-by-2 matrix.

Consider subspaces $H_a^1 = \{(a, b, 0, d) \in H^1(\rho)\}$, $H_{a'}^1 = \{(0, b, c, d) \in H^1(\rho)\}$, then by the previous arguments $\dim(H_a^1) = \dim(H_{a'}^1) = 1$ and $H^1(\rho) = H_a^1 \oplus H_{a'}^1$. We will show that they are orthogonal subspaces under A_G . For any $(a, b, 0, d) \in H_a^1(\rho)$, $(0, b', c', d') \in H_{a'}^1(\rho)$, because $(*, *)_G$ is G -invariant and $d^1(a, b, 0, d) = d^1(0, b', c', d') = 0$, we have

$$\begin{aligned} A_G([a, b, 0, d], [0, b', c', d']) &= \frac{1}{2i}(-(a, b')_G + (d, c')_G + (a, h_2b')_G - (h_2h_3h_4d, h_2h_3c')_G) \\ &= \frac{1}{2i}((b, b')_G + (d, b')_G - (d, b')_G - (d, d')_G - (h_2b, h_2b')_G \\ &\quad - (h_2h_3h_4d, h_2b')_G + (h_2h_3h_4d, h_2h_3h_4d')_G + (h_2h_3h_4d, h_2b')_G) \\ &= 0 \end{aligned}$$

In other words, A_G is diagonalized under $H^1(\rho) = H_a^1 \oplus H_{a'}^1$.

Now we show that the signature of A_G on H_a^1 is $(3 - \frac{\theta_{1a} + \theta_{2a}}{2\pi}, \frac{\theta_{2a}}{2\pi} - 1, \frac{\theta_{1a}}{2\pi} - 1)$, where $\theta_{1a} = \arg(\rho(g_1)) + \arg(\rho(g_2)) + \arg(\rho(g_3g_4))$, $\theta_{2a} = \arg(\rho(g_1)^{-1}) + \arg(\rho(g_2)^{-1}) + \arg(\rho(g_3g_4)^{-1})$. From (10) and (11) we know that

$$(12) \quad H_a^1 = \{t(\rho(g_2) - \rho(g_1^{-1}), \rho(g_1^{-1}) - 1, 0, \rho(g_2) - 1) : t \in D_\rho\}$$

If $\rho(g_2) = 0$, equation (12) becomes $H_a^1 = \{(t, -t, 0, 0) : t \in D_\rho\}$. If $\rho(g_1) = 0$, (12) becomes $H_a^1 = \{(t, 0, 0, -t) : t \in D_\rho\}$. In both cases the signature is $(1, 0, 0) = (3 - \frac{\theta_{1a} + \theta_{2a}}{2\pi}, \frac{\theta_{2a}}{2\pi} - 1, \frac{\theta_{1a}}{2\pi} - 1)$. If neither $\rho(g_1)$ nor $\rho(g_2)$ is 1, $\theta_{1a} + \theta_{2a} = 6\pi$, and θ_{1a} is either 2π or 4π . From (9) and (12) we know that A_G is positive on H_a^1 definite when $\theta_{1a} = 2\pi$ and negative definite on H_a^1 when $\theta_{1a} = 4\pi$, i.e. the signature of A_G on H_a^1 is $(3 - \frac{\theta_{1a} + \theta_{2a}}{2\pi}, \frac{\theta_{2a}}{2\pi} - 1, \frac{\theta_{1a}}{2\pi} - 1)$.

We can calculate the signature of A_G on $H_{a'}^1$ similarly. Because A_G is diagonalized under $H^1(\rho) = H_a^1 \oplus H_{a'}^1$, the n_0 , n_+ and n_- of A_G on $H^1(\rho)$ can be obtained by adding the n_0 , n_+ and n_- of A_G on H_a^1 and $H_{a'}^1$. \square

5. A SUBGROUP OF Γ

In this section we introduce a subgroup Γ_1 of Γ of finite index, which is easier to work with than Γ . In section 6, we will give a criteria for non-discreteness of the action of Γ by

analyzing the action of this subgroup of finite index.

There is a homomorphism $D : \mathbf{Aff} \rightarrow SL(2, \mathbb{Z})$ that sends an affine diffeomorphism to its derivative. Because elements of \mathbf{Aff} preserves Σ , $\ker(D)$ is finite. Consider two elements in Γ which are liftings of the horizontal and vertical Dehn twists of the pillowcase

$$\gamma_1 = s_{(g_1, g_2, g_3, g_4)} s_{(g_2, g_1, g_3, g_4)}$$

$$\gamma_2 = t_{(g_1, g_2, g_3, g_4)} s_{(g_2, g_3, g_4, g_1)} s_{(g_3, g_2, g_4, g_1)} t_{(g_1, g_2, g_3, g_4)}^{-1}$$

$D\gamma_1$ and $D\gamma_2$ generates the level 2 congruence subgroup of $SL(2, \mathbb{Z})$, hence the group generated by them is a subgroup of \mathbf{Aff} of finite order. From (5), (6) we have:

$$(13) \quad \gamma_1^*(a, b, c, d) = (g_1 g_2 a, b + a - g_1 a, c, d + g_1 a - g_1 g_2 a)$$

$$(14) \quad \gamma_2^*(a, b, c, d) = (a + g_2 b - g_2 g_3 b, g_2 g_3 b, c + b - g_2 b, d)$$

Denote the group generated by γ_1 and γ_2 as Γ_1 . When restricted to $H^1(\rho)$,

$$(15) \quad \gamma_1^*(a, b, c, d) = (\rho(g_1 g_2) a, b + a - \rho(g_1) a, c, d + \rho(g_1) a - \rho(g_1 g_2) a)$$

$$(16) \quad \gamma_2^*(a, b, c, d) = (a + \rho(g_2) b - \rho(g_2 g_3) b, \rho(g_2 g_3) b, c + b - \rho(g_2) b, d)$$

When $\rho = 1$ is the trivial representation, the Γ_1 action on $H^1(\rho)$ is trivial. When ρ is non-trivial, the Γ action on $H^1(\rho) = \mathbb{C}^2$ induces an action on \mathbb{CP}^1 under projectivization. The map on \mathbb{CP}^1 induced by γ_1 is parabolic if and only if $\rho(g_1 g_2) = 1$. Similarly, γ_2 is parabolic if and only if $\rho(g_2 g_3) = 1$. If they are not parabolic they are elliptic.

Furthermore, when $\rho(g_2) = 1$ and all other $\rho(g_j) \neq 1$, the Γ_1 action $H^1(\rho)$ is not semisimple. We can see this as follows: by equations (15) and (16), $(1, -1, 0, 0)$ is the only common eigenvector of γ_1^* and γ_2^* in $H^1(\rho)$, hence $H_a^1(\rho) = \{(t, -t, 0, 0)\}$ is the only 1-dimensional subspace of $H^1(\rho)$ invariant under Γ_1 , i.e. in this case the Γ_1 action on $H^1(\rho)$ is not semisimple. Furthermore, from the discussions in Section 6, 7 we know the Γ_1 action on $H^1(\rho)$ is semisimple in other cases. In other words, we have:

Proposition 5.1. *The Γ_1 action on $H^1(\rho)$ is not semisimple if and only if exactly one of the four complex numbers $\rho(g_1)$, $\rho(g_2)$, $\rho(g_3)$, $\rho(g_4)$ is 1.*

6. THE SPHERICAL CASE AND POLYHEDRAL GROUPS

The Hodge norm A_G on $H^1(\rho)$ induces a metric, hence a geometric structure on the projectivization $\mathbb{P}(H^1(\rho)) = \mathbb{CP}^1$ invariant under the Γ -action. When A_G is positive definite or negative definite, it induces a spherical structure on \mathbb{CP}^1 . When A_G has signature $(1, 0, 1)$, it induces a Euclidean structure on $\mathbb{CP}^1 - [0 : 1]$. When A_G has signature $(1, 1, 0)$, it induces a Euclidean structure on $\mathbb{CP}^1 - [1 : 0]$. Finally, when the signature of A_G is $(0, 1, 1)$, it induces a hyperbolic structure on a disc D in $\mathbb{P}H^1(\rho)$, which consists of the image $\{\alpha \in H^1(\rho) : A(\alpha, \alpha) > 0\}$. In this section we will describe the spherical case, and in the next section we will describe the remaining cases.

When A_G is positive definite or negative definite, the generators of Γ_1 , γ_1 and γ_2 , act as finite order rotations with different fixed points, and their orders are the orders of $\rho(g_1g_2)$ and $\rho(g_2g_3)$ in \mathbb{C}^* respectively, hence by the ADE classification [Dic59] of finite subgroups of $SO(3)$ we know that if both the orders of $\rho(g_1g_2)$ and $\rho(g_2g_3)$ are greater than 5 the action of Γ on $H^1(\rho)$ can not factor through a discrete group.

Example: Let G be the subgroup of $(\mathbb{Z}/120)^3$ spanned by $g_1 = (20, 0, 0)$, $g_2 = (0, 15, 0)$, $g_3 = (0, 0, 12)$, $g_4 = (100, 105, 108)$, $\mathbf{g} = (g_1, g_2, g_3, g_4)$, $M = M(G, \mathbf{g})$, $\rho(g_1) = e^{\pi i/3}$, $\rho(g_2) = e^{\pi i/4}$, $\rho(g_3) = e^{\pi i/5}$, $\rho(g_4) = e^{73i\pi/60}$. Then by Theorem 3.1 and Remark 1 $H^1(\rho)$ is invariant under \mathbf{Aff} with an invariant complement, by section 4 the Hodge form is positive definite on $H^1(\rho)$, and by the argument above the \mathbf{Aff} action on $H^1(\rho)$ is not discrete. In other words, M and the decomposition $H^1(M, \Sigma; \mathbb{C}) = H^1(\rho) \oplus (\bigoplus_{\rho' \neq \rho} H^1(\rho'))$ satisfy all the conditions mentioned in the beginning of section 1.

Furthermore, using [Cox73], we can list all possible 4-tuples $(\rho(g_1), \rho(g_2), \rho(g_3), \rho(g_4))$ such that the Γ_1 action on $\mathbb{P}(H^1(\rho)) = \mathbb{CP}^1$ factors through a finite group. Denote the arguments of $\rho(g_j)$ as $2t_j\pi$, $j = 1, \dots, 4$, then the Γ_1 action is finite if and only if t_1, t_2, t_3, t_4 is a permutation of one of the 4-tuples in the table below:

t_1	t_2	t_3	t_4	Group
$d/2n$	$d/2n$	$(n-d)/2n$	$(n-d)/2n$	Dihedral group
$1/12$	$1/4$	$1/4$	$5/12$	Tetrahedral group
$1/24$	$5/24$	$7/24$	$11/24$	Octahedral group
$1/60$	$11/60$	$19/60$	$29/60$	Icosahedral group
$1/6$	$1/6$	$1/6$	$1/2$	Tetrahedral group
$1/12$	$1/6$	$1/6$	$7/12$	Octahedral group
$1/30$	$19/30$	$1/6$	$1/6$	Icosahedral group
$1/30$	$3/10$	$3/10$	$11/30$	Icosahedral group
$1/20$	$3/20$	$7/20$	$9/20$	Icosahedral group
$1/15$	$2/15$	$4/15$	$8/15$	Icosahedral group
$1/10$	$3/10$	$3/10$	$3/10$	Icosahedral group
$1/10$	$1/10$	$7/30$	$17/30$	Icosahedral group
$1/10$	$1/10$	$1/10$	$7/10$	Icosahedral group
$7/60$	$13/60$	$17/60$	$23/60$	Icosahedral group
$1/6$	$1/6$	$7/30$	$13/30$	Icosahedral group
$(2n-d)/2n$	$(2n-d)/2n$	$(n+d)/2n$	$(n+d)/2n$	Dihedral group
$7/12$	$3/4$	$3/4$	$11/12$	Tetrahedral group
$13/24$	$19/24$	$17/24$	$23/24$	Octahedral group
$31/60$	$41/60$	$49/60$	$59/60$	Icosahedral group
$1/2$	$5/6$	$5/6$	$5/6$	Tetrahedral group
$5/12$	$5/6$	$5/6$	$11/12$	Octahedral group
$5/6$	$5/6$	$11/30$	$29/30$	Icosahedral group
$19/30$	$7/10$	$7/10$	$29/30$	Icosahedral group
$11/20$	$13/20$	$17/20$	$19/20$	Icosahedral group
$7/15$	$11/15$	$13/15$	$14/15$	Icosahedral group
$7/10$	$7/10$	$7/10$	$9/10$	Icosahedral group
$13/30$	$23/30$	$9/10$	$9/10$	Icosahedral group
$3/10$	$9/10$	$9/10$	$9/10$	Icosahedral group
$37/60$	$43/60$	$47/60$	$53/60$	Icosahedral group
$17/30$	$23/30$	$5/6$	$5/6$	Icosahedral group

Here d and n are positive integers, and the last column shows the discrete subgroup of $SO(3)$ it corresponds to.

Examples of M and ρ that satisfies the conditions in Section 1 can be built from any 4-tuple of positive rational numbers not on the above list that sum up to 1 or 3. For example, $(1/8, 1/8, 1/8, 5/8)$ is not on the list, so let $G = \mathbb{Z}/8$, $g_1 = g_2 = g_3 = 1$, $g_4 = 5$, $\rho(g_1) = \rho(g_2) = \rho(g_3) = e^{\pi i/4}$, $\rho(g_4) = e^{5\pi i/4}$ satisfies the conditions in Section 1. This is an abelian cover of flat pillowcase that satisfy the conditions in section 1 with the smallest number of squares.

7. THE HYPERBOLIC AND EUCLIDEAN CASES AND TRIANGLE GROUPS

In this section we complete the description of Γ_1 -action by describing the signature $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ cases as triangle groups. In the $(0, 1, 1)$ case, $\sum_j \arg(\rho(g_j)) = 4\pi$. Without losing generality we assume $\arg(\rho(g_1)) + \arg(\rho(g_2)) \geq 2\pi$, $\arg(\rho(g_2)) + \arg(\rho(g_3)) \geq 2\pi$.

By Remark 1 and Section 5, γ_1 acts on D as a rotation by $\arg(\rho(g_1g_2))$ when $\rho(g_1g_2) \neq 1$ and as a parabolic transform when $\rho(g_1g_2) = 1$, γ_2 acts on D as a rotation by $\arg(\rho(g_2g_3))$ when $\rho(g_2g_3) \neq 1$ and as a parabolic transform when $\rho(g_2g_3) = 1$. Furthermore, γ_1 and γ_2 generate an index 2 subgroup of a triangle group, and the angles of the triangle are $|\pi - \arg(\rho(g_1g_2))/2|$, $|\pi - \arg(\rho(g_2g_3))/2|$ and $|\pi - \arg(\rho(g_1g_3))/2|$. [Wri12] has calculated the Lyapunov exponents from the area of this triangle.

When the signature of A_G is $(1, 1, 0)$ or $(1, 0, 1)$, only one $\rho(g_j)$ is equal to 1. Without losing generality assume $\rho(g_2) = 1$, then $(a, b, c, d) \mapsto b/a$ sends $H^1(\rho)$ to $\overline{\mathbb{C}}$, and under this map Γ_1 acts on $\mathbb{C} = \overline{\mathbb{C}} - \{\infty\}$ as an index-2 subgroup of a Euclidean triangle group. The angles of the triangle are $\arg(\rho(g_1))/2$, $\arg(\rho(g_3))/2$ and $\arg(\rho(g_4))/2$ when

$$\arg(\rho(g_1)) + \arg(\rho(g_3)) + \arg(\rho(g_4)) = 2\pi$$

i.e. when the signature of A_G is $(1, 1, 0)$. The angles of the triangles are $\pi - \arg(\rho(g_1))/2$, $\pi - \arg(\rho(g_3))/2$ and $\pi - \arg(\rho(g_4))/2$ when

$$\arg(\rho(g_1)) + \arg(\rho(g_3)) + \arg(\rho(g_4)) = 4\pi$$

i.e. when the signature of A_G is $(1, 0, 1)$.

When two of the four $\rho(g_j)$ are equal to 1, then the Γ_1 action on $H^1(\rho)$ factors through a finite abelian group.

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